NORTH-HOLLAND

Invariants of Vector-Valued Bilinear and Sesquilinear Forms

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ABSTRACT

First steps in the algebraic invariant theory of vector-valued bilinear and sesquilinear forms are made. In particular, explicit formulas for generators of all invariant rational functions for such forms are derived. These formulas, and certain analogues, have applications to the geometry of Riemannian submanifolds, distributions, and CR structures.

1. INTRODUCTION

The second fundamental form at a point of a Riemannian submanifold is a symmetric bilinear vector-valued form; the obstruction to the integrability of a distribution on a manifold is measured by a skew-symmetric bilinear vector-valued form; the Levi form at a point of a CR manifold is a Hermitian sesquilinear vector-valued form. Algebraic invariants of these forms, evaluated at each point of the manifold, give rise to global geometric invariants. In Riemannian geometry and the theory of distributions, these new invariants should prove to be related to the standard invariants in interesting ways. In higher-codimensional CR geometry, there are no standard invariants: these new invariants are the best that are presently available.

Before geometric applications can be made, the basic algebraic invari-

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ant theory must be understood. Hence this paper. We describe a set of generators for the invariant rational functions of a bilinear (Theorem 1) or sesquilinear (Theorem 2) vector-valued form. Since the bilinear case is slightly easier, we examine it first (Section 3), and then adapt the argument to the sesquilinear case (Section 4). Specialization to symmetric, skew-symmetric, and Hermitian forms is straightforward. These two main sections are preceded by a summary of some basic facts and notation (Section 2) and followed by a brief discussion of several open questions (Section 5). Our principal results are given in terms of coordinates to facilitate explicit computations and geometric applications. Their proofs use standard multilinear algebra and very basic representation theory, along with some central results of classical invariant theory, which we restate in the precise form we need. No background in invariant theory is presupposed.

Although pairs of real symmetric or Hermitian forms are dealt with in standard texts, there does not seem to be any modern literature on vector-valued forms in general. The closest analogue is the invariant theory of $n \times n$ matrices, which has been well studied (e.g., see [2] and [10]), but different group actions are involved, and significant links between the two areas are not apparent. From the standpoint of classical invariant theory, the problem of finding invariants of vector-valued forms can be subsumed under the more general theory of combinants (see pp. 254–265 of [8] for details and further references). However, since the nineteenth-century literature on combinants is accessible only to those familiar with the classical techniques (especially the symbolic method), we shall not exploit this connection, except to mention that the approach used in this paper can easily be extended to combinants in general, should anyone care to resuscitate this classical topic.

2. SOME BASIC FACTS AND DEFINITIONS

(a) Invariant Theory

Let V be a vector space acted on linearly by the group G; the field of scalars K may be either $\mathbb{R}$ or $\mathbb{C}$. A function $f : V \to K$ is a *relative invariant* of weight χ if $f(gv) = \chi(g)f(v)$ for all $g \in G$ and $v \in V$, where χ is a K-valued abelian character of G (i.e., $\chi : G \to K - \{0\}$ is a homomorphism). If χ is the trivial character, then $f(gv) = f(v)$, and f is called an *absolute invariant.* When the weight is of no concern, we shall speak simply of an *invariant.*

The following facts are well known (e.g., see [1, pp. 5–9]):

(1) Every rational invariant is the quotient of polynomial invariants.

(2) Each homogeneous component of a polynomial invariant is itself invariant.

(3) The invariant homogeneous polynomials of degree r are determined by the invariant r-linear functions on the r-fold Cartesian product $V^{\times r} = V \times V \times \cdots \times V$. (For convenience, we shall abuse terminology and call such functions r-linear functions on V.)

(4) An invariant r-linear function on V is equivalent to an invariant linear function on the r-fold tensor product $V^{\otimes r} = V \otimes V \otimes \cdots \otimes V$.

The actions of G on $V^{\times r}$ and $V^{\otimes r}$ implicitly referred to in (3) and (4) are induced in the obvious way by the action of G on V:

$$g(v_1, v_2, \ldots, v_r) = (gv_1, gv_2, \ldots, gv_r) \quad \text{and}$$
$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_r.$$

Our goal is to find generators for the rational functions that arise in connection with the invariant theory of a bilinear or sesquilinear vector-valued form. In light of (1) and (2), it suffices to find, for each positive integer r, a spanning set for the vector space of invariant homogeneous polynomials of degree r. This is done in Theorems 1 and 2. The proofs of these theorems rely heavily on (3) and (4).

(b) Indicial Conventions

For any positive integer m, the permutation symbol $\varepsilon^{i_1 i_2 \cdots i_m}$ is defined as usual to equal 1 (-1) if $i_1, i_2, \ldots, i_m$ is an even (odd) permutation of $1, 2, \ldots, m$, and to equal 0 otherwise. We introduce a shorthand for products of d such symbols (where d is some positive integer). The product $\varepsilon^{i_1 \cdots i_m} \varepsilon^{i_{m+1} \cdots i_{2m}} \cdots \varepsilon^{i_{dm-m+1} \cdots i_{dm}}$ will be written $\varepsilon^I(m, dm, 1)$. The final 1 stands for the identity element of the permutation group S_{dm}. More generally, for any permutation $\sigma \in S_{dm}$ the product

$$\varepsilon^{i_{\sigma(1)} \cdots i_{\sigma(m)}} \varepsilon^{i_{\sigma(m+1)} \cdots i_{\sigma(2m)}} \cdots \varepsilon^{i_{\sigma(dm-m+1)} \cdots i_{\sigma(dm)}}$$

will be written $\varepsilon^I(m, dm, \sigma)$. The symbols $\varepsilon_{i_1 i_2 \cdots i_m}$ and $\varepsilon_I(m, dm, \sigma)$ are defined similarly.

The Einstein summation convention will always be in force: whenever an index occurs both as a subscript and as a superscript in the same term of an expression, summation is indicated.

3. BILINEAR FORMS

Let V and W be vector spaces of dimensions n and c respectively over the field of scalars K (which is either $\mathbb{R}$ or $\mathbb{C}$), and let G be the group $\mathrm{Aut}(V) \times \mathrm{Aut}(W)$. The bilinear maps from $V \times V$ to W constitute a vector space $\mathrm{Bil}(V, W)$ isomorphic to the vector space $V^* \otimes V^* \otimes W$. The obvious action of G on $V^* \otimes V^* \otimes W$ determines an equivalent linear action on $\mathrm{Bil}(V, W)$:

$$gb(x, y) = pb(a^{-1}x, a^{-1}y) \qquad \text{for all} \qquad \begin{aligned} &g = (a, p) \in G, \\ &b \in \mathrm{Bil}(V, W), \text{ and } x, y \in V. \end{aligned}$$

Another way to arrive at this action is to insist that gb be defined so that the following diagram commutes:

$$\begin{array}{ccc} & b & \\ V \times V & \longrightarrow & W \\ a \times a \downarrow & & \downarrow p \\ V \times V & \longrightarrow & W \\ & gb & \end{array}$$

The chief result of this section is Theorem 1, which describes generators for the invariants of this action of G on $\mathrm{Bil}(V, W)$.

A form $b \in \mathrm{Bil}(V, W)$ is *symmetric* [*skew-symmetric*] if $b = b^T$ $[-b^T]$, where $b^T(x, y) = b(y, x)$ for all $x, y \in V$. The symmetric [skew-symmetric] forms constitute a G-invariant irreducible subspace of $\mathrm{Bil}(V, W)$, denoted by $\mathrm{Sym}(V, W)$ [$\mathrm{Skew}(V, W)$]. The maps $\pi : b \to \frac{1}{2}(b + b^T)$ and $\pi' : b \to \frac{1}{2}(b - b^T)$ project $\mathrm{Bil}(V, W)$ onto $\mathrm{Sym}(V, W)$ and $\mathrm{Skew}(V, W)$ respectively; we see that

$$\mathrm{Bil}(V, W) = \mathrm{Sym}(V, W) \oplus \mathrm{Skew}(V, W).$$

By restriction, an invariant defined on $\mathrm{Bil}(V, W)$ determines an invariant on $\mathrm{Sym}(V, W)$. Moreover, every invariant on $\mathrm{Sym}(V, W)$ is obtainable in this way: if f is an invariant on $\mathrm{Sym}(V, W)$, then $f \circ \pi$ is an invariant on $\mathrm{Bil}(V, W)$ that restricts to f. Similar remarks apply to $\mathrm{Skew}(V, W)$. Consequently, once we have a set of generators for invariants of bilinear forms, we can obtain sets of generators for invariants of symmetric or skew-symmetric forms by restriction. (Of course, the restriction of a nontrivial invariant may vanish identically. However, in this paper we are interested primarily in completeness and defer all issues of economy.) Therefore, we proceed by studying invariants of bilinear forms.

The introduction of coordinates facilitates the description of invariants. Thus, let $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_c$ be bases of V and W, and let $e^1, e^2, \ldots, e^n$ and $f^1, f^2, \ldots, f^c$ be the dual bases of V^* and W^*. Each automorphism $a \in \mathrm{Aut}(V)$ determines a matrix $A = (A^j_k)$ in $\mathrm{Gl}(n, K)$ such that $ae_k = A^j_k e_j$. Similarly, each automorphism $p \in \mathrm{Aut}(W)$ determines a matrix $P = (P^j_k)$ in $\mathrm{Gl}(c, K)$ such that $pf_k = P^j_k f_j$. Finally, each W-valued bilinear form $b \in \mathrm{Bil}(V, W)$ determines a c-tuple of scalar bilinear forms $(b^1, b^2, \ldots, b^c)$, where $b^\alpha = f^\alpha \circ b$; in turn, each scalar form b^α determines an $n \times n$ matrix B^α, where $B^\alpha_{jk} = b^\alpha(e_j, e_k)$. In terms of these coordinates, the action of G on $\mathrm{Bil}(V, W)$ is given as follows: if $d = (a, p)b$ then

$$D^\alpha_{jk} = P^\alpha_\beta (A^{-1})^r_j (A^{-1})^s_k B^\beta_{rs}.$$

Using the preceding notation in conjunction with the indicial conventions of Section 2, we can describe a set of generators for the invariants of bilinear forms.

THEOREM 1 (First fundamental theorem for vector-valued bilinear forms). Let r be a positive integer.

(1) There exist nonzero homogeneous invariants of degree r on $\mathrm{Bil}(V, W)$ only if $2r$ is divisible by n and r is divisible by c.

(2) Suppose that $2r = nu$ and $r = cv$, where u and v are positive integers. Given permutations $\sigma \in S_{2r}$ and $\eta \in S_r$, define a polynomial f^σ_η on $\mathrm{Bil}(V, W)$ as follows:

$$f^\sigma_\eta(b) = \varepsilon^I(n, 2r, \sigma)\varepsilon_J(c, r, \eta) B^{j_1}_{i_1 i_2} B^{j_2}_{i_3 i_4} \cdots B^{j_r}_{i_{2r-1} i_{2r}}.$$

Each such polynomial is a relative invariant of weight χ, where $\chi : (a, p) \to (\det a)^{-2u}(\det p)^v$.

More generally, a homogeneous polynomial of degree r is a relative invariant if and only if it is a linear combination of the various polynomials f^σ_η. In particular, all relative invariants of degree r have weight χ.

Proof. If V and W are real vector spaces, then every $\mathrm{Aut}(V) \times \mathrm{Aut}(W)$-invariant polynomial on $\mathrm{Bil}(V, W)$ can be interpreted as an $\mathrm{Aut}(\mathbb{C} \otimes V) \times \mathrm{Aut}(\mathbb{C} \otimes W)$-invariant polynomial on $\mathrm{Bil}(\mathbb{C} \otimes V, \mathbb{C} \otimes W)$. Therefore, without loss of generality we may assume that V and W are both complex vector spaces. Since G acts linearly on $\mathrm{Bil}(V, W)$, the results outlined in Section 2 apply. In particular, the homogeneous invariants of degree r are determined by the linear invariants on the r-fold tensor product $\mathrm{Bil}(V, W)^{\otimes r}$. By elementary linear algebra, each such linear invariant corresponds to an invariant one-dimensional subspace of the r-fold tensor product $(\mathrm{Bil}(V,W)^*)^{\otimes r}$. Finally, since $\mathrm{Bil}(V,W) = V^* \otimes V^* \otimes W$, it suffices to describe all invariant one-dimensional subspaces of the space $X = M \otimes N$, where M is the $2r$-fold tensor product $V^{\otimes 2r}$ and N is the r-fold tensor product $(W^*)^{\otimes r}$.

Since the action of G on X is built up from the actions of $\mathrm{Aut}(V)$ on M and $\mathrm{Aut}(W)$ on N, the following specializations of well-known results are applicable:

(a) M $[N]$ has $\mathrm{Aut}(V)$-invariant $[\mathrm{Aut}(W)$-invariant$]$ one-dimensional subspaces if and only if $2r$ $[r]$ is divisible by n $[c]$.

(b) Suppose that $2r = nu$, where u is a positive integer, let σ be a permutation in S_{2r}, and let

$$v_\sigma = \varepsilon^I(n, 2r, \sigma) e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{2r}}.$$

The vector v_σ is relatively invariant: indeed, if $a \in \mathrm{Aut}(V)$ then $av_\sigma = (\det a)^{2u} v_\sigma$. More generally, a vector $v \in M$ is relatively invariant if and only if it is a linear combination of the various v_σ.

(c) Suppose that $r = cv$, where v is a positive integer, let η be a permutation in S_r, and let

$$w^\eta = \varepsilon_J(c, r, \eta) f^{j_1} \otimes f^{j_2} \otimes \cdots \otimes f^{j_r}.$$

The vector w^η is relatively invariant: indeed, if $p \in \mathrm{Aut}(W)$ then $pw^\eta = (\det p)^{-v} w^\eta$. More generally, a vector $w \in N$ is relatively invariant if and only if it is a linear combination of the various w^η.

(d) M and N can be decomposed into irreducible subspaces

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_s \quad \text{and} \quad N = N_1 \oplus N_2 \oplus \cdots \oplus N_t$$

for some positive integers s and t.

The first three statements follow easily from standard treatments of one-dimensional tensor representations of the general linear group (see [1, p. 21] or [13, Chapter IV]). They also can be derived from the fundamental theorem of invariant theory for the general linear group, which is proved in Chapter II of [13] by means of the Capelli identities, and in §16 and §17 of [5] by linear algebra. The final statement follows from the fact that every polynomial representation of the general linear group is completely reducible (see §41 and §42 of [14]).

Armed with these statements, we proceed to the study of G-invariant one-dimensional subspaces of $X = M \otimes N$. Suppose that Y is such a subspace. It follows from (d) that $X = \bigotimes(M_i \otimes N_j)$; let π_{ij} denote the projection of X onto $M_i \otimes N_j$. Each subspace $M_i \otimes N_j$ is G-irreducible, since M_i is $\mathrm{Aut}(V)$-irreducible and N_j is $\mathrm{Aut}(W)$-irreducible, and the vector spaces are all complex (see [9, p. 45]). Consequently, by Schur's lemma, a given projection π_{ij} either vanishes identically on Y or induces an isomorphism of Y with $M_i \otimes N_j$. Since Y is not the trivial subspace, not all of the π_{ij} can vanish. Therefore, Y is isomorphic to at least one of the subspaces $M_i \otimes N_j$. Since Y is one-dimensional, both M_i and N_j must be one-dimensional. Therefore, (a) implies that $2r$ is divisible by n and r is divisible by c. Thus, we have proved the following statement:

(I) X has an invariant one-dimensional subspace only if $2r$ is divisible by n and r is divisible by c.

Now suppose that $2r = nu$ and $r = cv$, let $T = \{(i, j) \mid \dim M_i = 1$ and $\dim N_j = 1\}$, and let $X_0 = \oplus_{(i,j) \in T} M_i \otimes N_j$. Clearly, $Y \subseteq X_0$. Moreover, it follows from (b) and (c) that the various vectors $v_\sigma \otimes w^\eta$ span X_0, and also that if $g = (a, p)$ is an element of G, then g acts on X_0 as multiplication by $(\det a)^{2u}(\det p)^{-v}$. Therefore, *every* one-dimensional subspace of X_0 must be invariant. We summarize what we have just proved as follows:

(II) Suppose that $2r = nu$ and $r = cv$. A vector $x \in X$ is relatively invariant if and only if it is a linear combination of the various vectors $v_\sigma \otimes w^\eta$, where σ and η range through S_{2r} and S_r respectively. Moreover, if x is relatively invariant and $g = (a, p)$ is an element of G, then $gx = (\det a)^{2u}(\det p)^{-v}x$.

Statements (I) and (II), translated into the language of multilinear maps on $\mathrm{Bil}(V, W)$ by means of isomorphisms discussed earlier [such as the one between $V^* \otimes V^* \otimes W$ and $\mathrm{Bil}(V, W)$], read as follows:

(I′) There exist invariant r-linear maps on $\mathrm{Bil}(V, W)$ only if $2r$ is divisible by n and r is divisible by c.

(II′) Suppose that $2r = nu$ and $r = cv$. Given permutations $\sigma \in S_{2r}$ and $\eta \in S_r$, define an r-linear map L^σ_η on $\mathrm{Bil}(V, W)$ as follows:

$$L^\sigma_\eta(b_1, b_2, \ldots, b_r) = \varepsilon^I(n, 2r, \sigma)\varepsilon_J(c, r, \eta)(b_1)^{j_1}_{i_1 i_2}(b_2)^{j_2}_{i_3 i_4} \cdots (b_r)^{j_r}_{i_{2r-1} i_{2r}}.$$

Each such map is a relative invariant of weight χ. More generally, an r-linear map is a relative invariant if and only if it is a linear combination of the various maps L^σ_η.

Finally, parts (1) and (2) of the theorem follow from (I′) and (II′) if the various arguments $b_1, b_2, \ldots, b_r$ are all replaced by a single $b \in \mathrm{Bil}(V, W)$. ∎

EXAMPLE. If V is two-dimensional and W is one-dimensional ($n = 2$

and $c = 1$), a simple first-degree invariant ($r = 1$) can be obtained by letting σ be the identity permutation in S_2 (there is no freedom in choosing η, since S_1 has only one element). Then

$$\begin{aligned} f_\eta^\sigma(b) &= \varepsilon^I(2,2,\mathrm{id})\varepsilon_J(1,1,\mathrm{id})B^{j_1}_{i_1 i_2} = \varepsilon^{i_1 i_2}\varepsilon_{j_1} B^{j_1}_{i_1 i_2} \\ &= \varepsilon^{i_1 i_2} B^1_{i_1 i_2} = B^1_{12} - B^1_{21}. \end{aligned}$$

This invariant vanishes in the symmetric case, but is nontrivial in the skew-symmetric case.

To obtain a second-degree invariant($r = 2$), the most obvious choice is to let σ and η be the identity permutations in S_4 and S_2 respectively; a direct calculation shows that this invariant is the square of the preceding invariant. A more interesting invariant results by letting σ be the transposition (23) in S_4:

$$\begin{aligned} f_\eta^\sigma(b) &= \varepsilon^I(2,4,(23))\varepsilon_J(1,2,\mathrm{id})B^{j_1}_{i_1 i_2} B^{j_2}_{i_3 i_4} = \varepsilon^{i_1 i_3}\varepsilon^{i_2 i_4}\varepsilon_{j_1}\varepsilon_{j_2} B^{j_1}_{i_1 i_2} B^{j_2}_{i_3 i_4} \\ &= \varepsilon^{i_1 i_3}\varepsilon^{i_2 i_4} B^1_{i_1 i_2} B^1_{i_3 i_4} = B^1_{11}B^1_{22} - B^1_{12}B^1_{21} - B^1_{21}B^1_{12} + B^1_{22}B^1_{11}; \end{aligned}$$

this is, of course, just twice the determinant of the matrix B^1.

It is worth noting here that for applications to Riemannian geometry, V and W are real vector spaces with fixed inner products, and one is interested not merely in $\mathrm{Aut}(V) \times \mathrm{Aut}(W)$-invariants, but in the larger class of $O(V) \times O(W)$-invariants, where $O(V)$ $[O(W)]$ is the group of isometries of V $[W]$. A theorem analogous to Theorem 1 can be proved by the same technique, replacing the classical theory of invariants of the general linear group by the equally classical theory of invariants of the orthogonal group. However, the formulas are somewhat complicated, and their derivation is perhaps best left to a paper on Riemannian geometry.

4. SESQUILINEAR FORMS

Before defining a vector-valued sesquilinear form, we must first define conjugation on an abstract complex vector space. Let U be a complex vector space. A *conjugation* on U is a real linear automorphism C that satisfies the following conditions:

(i) C is an involution—i.e., C^2 is the identity map.

(ii) C is complex antilinear—i.e., $C(\lambda u) = \overline{\lambda} C(u)$ for all $\lambda \in \mathbb{C}$ and $u \in U$.

For convenience, we shall often write $\bar{u}$ instead of $C(u)$. The eigenvalues of C are 1 and -1; corresponding eigenvectors are called real and imaginary respectively; the eigenspaces are denoted $\operatorname{Re} U$ and $\operatorname{Im} U$. The projections $\pi_R : U \to \operatorname{Re} U$and $\pi_I : U \to \operatorname{Im} U$ defined by

$$u \to \frac{1}{2}(u + \bar{u}) \quad \text{and} \quad u \to \frac{1}{2}(u - \bar{u})$$

establish the decomposition $U = \operatorname{Re} U \oplus \operatorname{Im} U$, where the summands are real (but not complex) vector spaces.

Let V and W be complex vector spaces of dimensions n and c respectively; moreover, let W be equipped with a conjugation. A real bilinear map $b : V \times V \to W$ is *sesquilinear* if $b(\lambda x, y) = \bar{\lambda} b(x, y)$ and $b(x, \lambda y) = \lambda b(x, y)$ for all $x, y \in V$ and $\lambda \in \mathbb{C}$; it is *Hermitian* [*skew-Hermitian*] if it is sesquilinear and $b(y, x) = \overline{b(x, y)}$ $[-\overline{b(x, y)}]$ for all $x, y \in V$. Let $\operatorname{Sesq}(V, W)$, $\operatorname{Herm}(V, W)$, and $\operatorname{Skew}(V, W)$ denote the spaces of sesquilinear, Hermitian, and skew-Hermitian forms.

Clearly, $\operatorname{Sesq}(V, W)$ is a complex vector space. Given a sesquilinear form s, define a new sesquilinear form s^* by requiring $s^*(x, y) = \overline{s(y, x)}$. The resulting map $s \to s^*$ is a conjugation on $\operatorname{Sesq}(V, W)$; the real and imaginary subspaces are $\operatorname{Herm}(V, W)$ and $\operatorname{Skew}(V, W)$. Therefore,

$$\operatorname{Sesq}(V, W) = \operatorname{Herm}(V, W) \oplus \operatorname{Skew}(V, W) \tag{1}$$

with corresponding projections $\pi : s \to \frac{1}{2}(s + s^*)$ and $\pi' : s \to \frac{1}{2}(s - s^*)$. Let $\operatorname{Aut}(V)$ and $\operatorname{Aut}(W)$ denote the groups of complex automorphisms of V and W. The product group $G = \operatorname{Aut}(V) \times \operatorname{Aut}(W)$ acts linearly on $\operatorname{Sesq}(V, W)$ by

$$\begin{aligned} gs(x, y) = ps(a^{-1}x, a^{-1}y) \quad \text{for all } g &= (a, p) \in G, \\ & s \in \operatorname{Sesq}(V, W), \text{ and } x, y \in V. \end{aligned}$$

Unfortunately, the decomposition (1) is not preserved by this action. However, it is preserved by the induced action of the subgroup $G_{\mathbb{R}} = \operatorname{Aut}(V) \times \operatorname{Aut}_{\mathbb{R}}(W)$, where $\operatorname{Aut}_{\mathbb{R}}(W)$ comprises all automorphisms of W that commute with the conjugation on W (real automorphisms for short).

By mimicking our earlier restriction argument for symmetric forms, we see that even if one is interested primarily in the invariant theory of Hermitian forms, it is reasonable to proceed by way of $G_{\mathbb{R}}$-invariants of general sesquilinear forms. [Since multiplication by i induces an isomorphism between $\operatorname{Herm}(V, W)$ and $\operatorname{Skew}(V, W)$, skew-Hermitian forms require no separate consideration.]

As in the bilinear case, coordinates are necessary. Thus, let $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_c$ be bases of V and W, and let $e^1, e^2, \ldots, e^n$ and $f^1, f^2, \ldots, f^c$ be the dual bases of V^* and W^*. Moreover, assume that each vector f_j is real: $f_j = \bar{f}_j$. Each automorphism $a \in \mathrm{Aut}(V)$ determines a matrix $A \in \mathrm{Gl}(n, \mathbb{C})$ such that $ae_k = A^j_k e_j$. Similarly, each automorphism $p \in \mathrm{Aut}(W)$ determines a matrix $P \in \mathrm{Gl}(c, \mathbb{C})$ such that $pf_k = P^j_k f_j$, and $p \in \mathrm{Aut}_{\mathbb{R}}(W)$ if and only if $p \in \mathrm{Gl}(c, \mathbb{R})$. Finally, each W-valued sesquilinear form $s \in \mathrm{Sesq}(V, W)$ determines a c-tuple of scalar sesquilinear forms $(s^1, s^2, \ldots, s^c)$, where $s^\alpha = f^\alpha \circ s$; in turn, each scalar form s^α determines a complex $n \times n$ matrix S^α, where $S^\alpha_{jk} = s^\alpha(e_j, e_k)$. In terms of these coordinates, the action of G on $\mathrm{Sesq}(V, W)$ is given as follows: if $t = (a, p)s$ then

$$T^\alpha_{jk} = P^\alpha_\beta (\bar{A}^{-1})^u_j (A^{-1})^v_k \, S^\beta_{uv}.$$

THEOREM 2 (First fundamental theorem for vector-valued sesquilinear forms). Let r be a positive integer.

(1) There exist nonzero homogeneous invariants of degree r on $\mathrm{Sesq}(V, W)$ only if r is divisible by both n and c.

(2) Suppose that $r = nu = cv$, where u and v are positive integers. Given permutations $\sigma, \tau, \eta \in S_r$, define a polynomial $f^{\sigma\tau}_\eta$ on $\mathrm{Sesq}(V, W)$ as follows:

$$f^{\sigma\tau}_\eta(s) = \varepsilon^I(n, r, \sigma)\varepsilon^J(n, r, \tau)\varepsilon_K(c, r, \eta) S^{k_1}_{i_1 j_1} S^{k_2}_{i_2 j_2} \cdots S^{k_r}_{i_r j_r}.$$

Each such polynomial is a relative invariant of weight χ, where

$$\chi : (a, p) \to (\det a)^{-u} (\det \bar{a})^{-u} (\det p)^v.$$

More generally, a homogeneous polynomial of degree r is a relative invariant if and only if it is a linear combination of the various polynomials $f^{\sigma\tau}_\eta$. In particular, all relative invariants of degree r have weight χ.

Proof. It is easy to verify that a polynomial on $\mathrm{Sesq}(V, W)$ is $G_{\mathbb{R}}$-invariant if and only if it is G-invariant. For technical convenience, we work with the full group G. In the proof of Theorem 1, a key step was the identification of $\mathrm{Bil}(V, W)$ with $V^* \otimes V^* \otimes W$, which facilitated the application of classical results on tensor invariants. Here, a similar identification of $\mathrm{Sesq}(V, W)$ with an appropriate tensor product is fundamental; we summarize the requisite elementary material on complex structures, using the notation of [7, Vol. 2, p. 117].

The n-dimensional complex vector space V can be viewed as a $2n$-dimensional real space; multiplication by i determines a complex structure J on V—that is, J is a real automorphism with the property that $-J^2$ is the identity map. The map J can be extended by complex linearity to the $2n$-dimensional complex vector space $\mathbb{C}\otimes V$. The eigenvalues of J are i and $-i$, the corresponding eigenspaces are denoted $V^{1,0}$ and $V^{0,1}$, and $\mathbb{C}\otimes V = V^{1,0}\oplus V^{0,1}$. Dually, the transpose of J is a complex linear automorphism of $(\mathbb{C}\otimes V)^*$ with eigenvalues i and $-i$ and eigenspaces $V_{1,0}$ and $V_{0,1}$; moreover, $(\mathbb{C}\otimes V)^* = V_{1,0}\oplus V_{0,1}$. The space $\mathbb{C}\otimes V$ has a natural conjugation with the property that $\overline{\lambda\otimes v} = \overline{\lambda}\otimes v$ for all $\lambda\in\mathbb{C}$ and $v\in V$. This conjugation induces a real isomorphism of $V^{1,0}$ and $V^{0,1}$. Viewed as a complex vector space, V is naturally isomorphic to $V^{1,0}$ by an isomorphism that sends x to $\frac{1}{2}(x-iJx)$. There are uniquely defined actions of $\mathrm{Aut}(V)$ on $V^{1,0}$ and $V^{0,1}$ which make the preceding isomorphisms $\mathrm{Aut}(V)$-equivariant. If $z_1,\dots,z_n$ and $\overline{z}_1,\dots,\overline{z}_n$ are the bases of $V^{1,0}$ and $V^{0,1}$ corresponding to the basis $e_1,\dots,e_n$ of V, and $a\in\mathrm{Aut}(V)$, then $az_k = A^j_k z_j$ and $a\overline{z}_k = \bar{A}^j_k\overline{z}_j$.

The verification of (1) is now straightforward. Indeed, since V is isomorphic to $V^{1,0}$, it follows that $\mathrm{Sesq}(V,W)$ is isomorphic to $\mathrm{Sesq}(V^{1,0},W)$. Moreover, if the map $B: V^{1,0}\times V^{1,0}\to W$ is sesquilinear, then the map $B': V^{0,1}\times V^{1,0}\to W$ defined by the equation $B'(x,y) = B(\overline{x},y)$ is bilinear; the correspondence $B\to B'$ sets up an isomorphism between $\mathrm{Sesq}(V^{1,0},W)$ and $\mathrm{Bil}(V^{0,1}\times V^{1,0},W)$. Finally, by the definition of tensor product, $\mathrm{Bil}(V^{0,1}\times V^{1,0},W)$ is isomorphic to $(V^{0,1})^*\otimes(V^{1,0})^*\otimes W$, which in turn is isomorphic to $V_{0,1}\otimes V_{1,0}\otimes W$.

We now reason as in the proof of Theorem 1. First, we show that it suffices to describe all G-invariant subspaces of $X = M\otimes N$, where M is the $2r$-fold tensor product $(V^{0,1})^{\otimes r}\otimes(V^{1,0})^{\otimes r}$ and N is the r-fold tensor product $(W^*)^{\otimes r}$. Next, we consider the actions of $\mathrm{Aut}(V)$ and $\mathrm{Aut}(W)$ on M and N respectively, and note that the following statements are specializations of well-known results. (For references, see the proof of Theorem 1.)

(a) M [N] has $\mathrm{Aut}(V)$-invariant [$\mathrm{Aut}(W)$-invariant] one-dimensional subspaces if and only if r is divisible by n [c].

(b) Suppose that $r = nu$, let σ and τ be permutations in S_r, and let

$$\bar{v}_\sigma = \varepsilon^I(n,r,\sigma)\bar{z}_{i_1}\otimes\bar{z}_{i_2}\otimes\cdots\otimes\bar{z}_{i_r} \quad\text{and}\quad v_\tau = \varepsilon^J(n,r,\tau)z_{j_1}\otimes_{j_2}\otimes\cdots\otimes z_{j_r}.$$

The vector $\overline{v}_\sigma\otimes v_\tau$ is relatively invariant: indeed, $a(\overline{v}_\sigma\otimes v_\tau) = (\det a)^u(\det\overline{a})^u\overline{v}_\sigma\otimes v_\tau$ for all $a\in\mathrm{Aut}(V)$. More generally, a vector $x\in M$ is relatively invariant if and only if it is a linear combination of the various vectors $\overline{v}_\sigma\otimes v_\tau$.

(c) Suppose that $r = cv$, let η be a permutation in S_r, and let

$$w^\eta = \varepsilon_K(c, r, \eta) f^{k_1} \otimes f^{k_2} \otimes \cdots \otimes f^{k_r}.$$

The vector w^η is relatively invariant: indeed, if $p \in \mathrm{Aut}(W)$ then $pw^\eta = (\det p)^{-v} w^\eta$. More generally, a vector $y \in N$ is relatively invariant if and only if it is a linear combination of the various w^η.

(d) M and N can be decomposed into irreducible subspaces:

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_s \quad \text{and} \quad N = N_1 \oplus N_2 \oplus \cdots \oplus N_t$$

for some positive integers s and t. The rest of the proof proceeds as in Theorem 1.

■

5. CONCLUSION

As noted in the introduction, the algebraic invariant theory of vector-valued forms has geometric ramifications. Theorem 2 is used in [6] to study CR geometry. Applications of Theorem 1 (and its orthogonal analogue) to the study of distributions and Riemannian geometry remain to be explored.

Unfortunately, a great deal of computation is required. In part, this is because our theorems describe spanning sets for vector spaces of invariant homogeneous polynomials. One possible improvement would be to refine these spanning sets into linear bases. Swanson [12] has written a computer program in C that produces such a basis when given specific values of r, n, and c. Alternatively, one can describe these bases more elegantly (and find a formula for their dimension) if one obtains the one-dimensional invariant subspaces on which our proofs are based by using Young diagrams rather than permutation symbols. The expert will undoubtedly prefer this approach, and find it easy to modify our proofs accordingly; the interested novice can learn the requisite preliminaries in [3].

Much greater simplification would result from a further development of the algebraic theory. In particular, one would like to have formulas for a minimal set of ring generators of the ring of polynomial invariants. One possible way of obtaining such formulas is first to find a bound on the degree of the generators, and then to use some type of Gröbner-basis argument. Also, one would like to have a second fundamental theorem which describes the relations satisfied by these generators. We suspect that Hilbert's techniques [4], recently explained in modern language in

[11], will apply. Moreover, one would like all of these results explicitly for symmetric, skew-symmetric, and Hermitian forms, as well as for bilinear and sesquilinear forms.

Finally, there remains the fundamental question of how to tell when two forms are equivalent. While algebraic invariants clearly provide necessary conditions, it appears unlikely that they are sufficient. Additional work on the Hermitian case is given in [6].

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